Evaluation Of Phytochemical Compounds Present in Some Medicinal Plants

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ABSTRACT

Medicinal plants are widely used in the treatment of several diseases. This is because of diverse pharmacological activities, shown by these plants such as antimicrobial, antidiabetic, antihypertensive, antidiarrheal, anthelmintic and several other therapeutic effects shown by these plants. The therapeutic potential of medicinal plants is determined by their phytochemical composition, either individually or in combination. Alkaloids, phenolics, tannins, steroids, glycosides and terpenes are important phytochemicals. The pharmacological properties of plants can be evaluated through the identification of these phytochemicals. Although modern analytical techniques are now available for phytochemical analysis, traditional qualitative tests continue to be widely used for preliminary phytochemical screening of medicinal plants.

Keywords : ***Medicinal Plants, Phytochemical compounds, Qualitative tests, Secondary metabolites.***

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Introduction:

Medicinal plants are now getting attention because they show many benefits especially in the field of medicine and pharmacology. Phytochemicals are natural bioactive compounds found in plants such as vegetables, fruits, flowers, leaves, roots and medicinal herbs. These phytochemicals work with food and fiber as antioxidants and as a defense mechanism against diseases (Kumar et. al., 2021). Phytochemical screening and antioxidant potential of selected medicinal plants using different solvents. Plant Sci. Today, 8(4): 824–832. Phytochemicals are divided into primary and secondary constituents based on their function in plant metabolism. The main components are common sugars, amino acids, protein and chlorophyll, the secondary components are alkaloids, terpenoids and phenolic compounds and many others.

The medicinal value of these plants is in its bioactive phytochemical constitution, which has a physiological effect on the human body (Akinmoladuh et al., 2007; Ahmad et. al., 2021). Most important and common phytoconstituents are flavonoids, alkaloids, tannins, terpenoids, Saponins, Phenolic, compound. These natural compounds are the basis of modern medicine prescribed nowadays (Edenga et. al., 2005).

Five important medicinal plants available in the Botanical garden of COCAS, Patliputra University were studied. Catharanthus roseus, Hibiscus rosa-sinesis, Moringa oleifera, Nycthanthes arbor-tritis, Phyllanthus amarus were the experimental plants. These were easily available and reports say that they are being utilized in traditional medicine. By characterizing the phytochemicals present in these plants they can be analysed and utilised by common men for medicine and mass propagation.

Materials and Methods:

Collection of plant samples: The plant materials were available in the Botanical garden of COCAS. Some materials were also brought from local markets and planted in the Botanical garden for studies.

The plant parts were properly washed in tap water and then rinsed in distilled water. These were then air dried in shade. The plant material was grinded in a mixer and the powder was kept in air tight containers. These were labelled and protected from direct sunlight until analysed.

Preparation of aqueous extract of plant samples

The aqueous extract of each plant sample was prepared by soaking 10 g of powdered samples in 200 ml of distilled water for 12 h. The extracts were then filtered with Whatman filter paper.

Phytochemical analysis:

Chemical tests were conducted on the aqueous extract of each plant sample, as well as of the stored powdered form of the plant samples using standard methods Edeoga et. al. (2005).

Qualitative Phytochemical analysis

Detection of Alkaloids:

Extracts were dissolved individually in dilute hydrochloric acid and filtered. The filtrates were used to test the presence of alkaloids. Two tests were done for alkaloid detection.

Mayer's test : Filtrates were treated with Mayer's reagent. Formation of a yellow cream precipitate indicates the presence of alkaloids.

Wagner's test : Filtrates were treated with Wagner's reagent. Formation of brown/reddish brown precipitate indicates the presence of alkaloids.

Detection of Terpenoids:

Salkowski's Test : 5 mg of the extract of the leaves, flowers and seeds was mixed with 2 ml of chloroform and concentrated H_2SO_4 (3ml) was carefully added to form a layer. An appearance of reddish brown colour in the inner face indicated the presence of terpenoids.

Detection of Phenols:

Ferric chloride test : 10 mg extracts were treated with few drops of ferric chloride solution. Formation of bluish black colour indicates the presence of phenol.

Lead acetate test : 10 mg extracts was treated with few drops of lead acetate solution. Formation of yellow colour precipitate indicates the presence of phenol.

Detection of Tannins:

A small quantity of extract was mixed with water and heated in a water bath. The mixture was filtered and ferric chloride was added to the filtrate. A dark green colour was formed. It indicates the presence of tannins.

Plants	Phenols	Tannin	Alkaloids	Saponins	Terpenoids
Catharanthus roseus	+	-	+	-	+
H. rosa-sinesis	+	+	+	+	+
Moringa oleifera	+	+	+	+	+
Nycthanthes arbor-tristis	+	+	+	+	+
Phyllanthus amarus	+	+	+	+	+

Table-1. Qualitative analysis of phytochemical constituents. (Presence of phytochemical constituents: +; Absence of phytochemical constituents : -)

Results And Discussion:

Qualitative Analysis : Qualitative analysis carried out on these plants showed the presence of phytochemical constituents. Table-1. shows that Phenols, Alkaloids and Terpenoids are present in all these plants. Catharanthus roseus shows the absence of tannins and saponins. H rosa-sinesis, Moringa oleifera, Nycthanthes arbor- tristis and Phyllanthus amarus shows the presence of tannins and saponins along with phenols, alkaloids and terpinoids.

The present analysis of phytochemicals in Catharanthus roseus, Hibiscus rosa-sinensis, Moringa oleifera, Nyctanthes arbor-tristis, and Phyllanthus amarus highlights the medicinal significance of these plants and supports their traditional and modern therapeutic applications. The detection of diverse bioactive constituents such as alkaloids, phenolics, flavonoids, tannins, saponins, glycosides, and terpenoids confirms their role as potential sources of pharmacologically active compounds, as previously emphasized by several researchers (Farnsworth and Morris, 1976; Gordon and David, 2001; Foye et al., 2008).

The findings are compatible with earlier studies reporting that the biological activity of medicinal plants is closely linked to their phytochemical composition (Harborne, 1973; Evans, 1966; Kokate et al., 2004). The choice of extraction solvent significantly influences the recovery of phytochemicals, as solvents differ in polarity and extraction efficiency This is a fact well documented in pharmacognostic literature (Brain and Turner, 1975; Krishnaiah et al., 2009). Similarly, the application of appropriate qualitative tests is essential for reliable phytochemical screening, and the use of more than one test enhances accuracy and reproducibility of results (Harborne, 1973; Kokate et al., 2004).

Furthermore, preliminary phytochemical screening serves as a scientific foundation for the targeted isolation, characterization, and validation of bioactive compounds, thereby facilitating advanced pharmacological and biochemical investigations (Geissman, 1963; Gordon and David, 2001). The present study also reinforces the importance of medicinal plants in drug discovery and public health, as recognized by the World Health Organization (WHO, 1985), (World Health Organization, 2019) and supports continued research into plant-based therapeutics.

Overall, the phytochemical evaluation of these selected medicinal plants not only validates their traditional use but also provides a strong basis for future studies aimed at developing novel plant-derived drugs with improved efficacy and safety.

Conclusion :

The analysis of phytochemicals in Catharanthus roseus, H. rosa-sinesis, Moringa oleifera, Nycthanthes arbor- tristis, Phyllanthus amarus is very important and the possible medicinal utilities of the plant. It is also helpful to determine the active principles responsible for the known biological activities exhibited by the plants. Further, it provides the base for targeted isolation of compounds and to perform more precise investigations. Extraction of a phytochemical from the plant material is mainly dependent on the type of solvents used. Similarly, the test applied for phytochemical analysis determines the presence or absence of a phytochemical in the sample. Hence, two or more different tests should be performed for more accurate results.

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Extensions of Inner Product Structures in Hilbertizable Banach Spaces

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ABSTRACT

Banach spaces and Hilbert spaces occupy a central position in functional analysis due to their wide-ranging theoretical importance and practical applications. While every Hilbert space is a Banach space, the converse is not true, as general Banach spaces do not necessarily admit an inner product structure. However, a special class of Banach spaces, known as Hilbertizable Banach spaces, can be equipped with an equivalent norm induced by an inner product, thereby allowing them to inherit the geometric and analytical properties of Hilbert spaces. This research article investigates the problem of extending inner product structures within such Hilbertizable Banach spaces. The study focuses on the conditions under which an inner product defined on a subspace, or obtained via an isomorphic mapping to a Hilbert space, can be consistently extended to the entire Banach space. Central concepts such as the parallelogram law, polarization identity, L-semi-inner products, and isometric isomorphisms are examined to establish criteria for Hilbertizability and extension. The article also discusses key characterizations due to Kwapień and Lindenstrauss–Tzafriri, highlighting their role in identifying Hilbert space structures. Through illustrative examples, including Sobolev spaces and applications in operator theory, the paper demonstrates the theoretical and practical significance of extending inner product structures. The results provide a unified framework for understanding how Hilbert space techniques can be effectively applied within broader Banach space settings.

Keywords: ***Banach, Hilbert Space; Hilbertizable Banach , Inner Product, Parallelogram,Functional Analysis***

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1. Introduction

Banach spaces and Hilbert spaces stand as two pivotal structures in the field of functional analysis. A Banach space is defined as a complete normed vector space, meaning that every Cauchy sequence in the space converges to an element within the space, with the norm providing a measure of vector magnitude and distance. On the other hand, an inner product space is a vector space equipped with an inner product that not only defines geometric notions such as lengths and angles but also induces a norm. When this inner product space is complete with respect to the induced norm, it becomes a Hilbert space.

Many Banach spaces are not inherently inner product spaces; however, some Banach spaces are "Hilbertizable." This term refers to Banach spaces that are isomorphic to Hilbert spaces in the sense that there exists an equivalent norm resulting from an inner product that endows the space with Hilbert space properties. This article investigates the extensions of inner product structures within these Hilbertizable Banach spaces. In other words, we analyze the conditions under which an inner product defined on a subspace or an approximate inner product concept, such as the L-semi-inner product, can be extended to yield a full inner product structure on the entire space. Such extensions have profound implications for both theoretical research and practical applications in areas such as differential equations, quantum mechanics, and approximation theory.

The discussion that follows is anchored in central results from functional analysis. We rely on fundamental properties like the parallelogram identity as a necessary and sufficient condition for the existence of an inner product norm. Additionally, we examine several characterizations of Hilbertizable Banach spaces, such as those provided by the works of Kwapieñ, and Lindenstrauss and Tzafriri. Our goal is to outline the theoretical framework necessary for extending inner product structures in Banach spaces that admit a Hilbert space structure, providing insights into both existing methodologies and open research problems.

2. Preliminaries

2.1 Banach Spaces and Normed Vector Spaces

A **Banach space** is a normed vector space in which every Cauchy sequence converges to a limit that is contained within the space. This property of completeness is crucial for many techniques in functional analysis, as it allows the use of limit processes and ensures stability under infinite operations[2]. In the context of Banach spaces, the norm provides a direct measure of the size or length of elements (vectors) and induces the metric $d(x,y)=\|x-y\|$

2.2 Inner Product Spaces and Hilbert Spaces

An **inner product space** is a vector space V over the real or complex numbers equipped with an inner product $\langle\cdot,\cdot\rangle$. This inner product must satisfy the following properties for all vectors $x,y,z\in v$ and all scalars a,b:

- **Linearity in the first argument**: $\langle ax+by,z\rangle = a\langle x,z\rangle + = \langle y,z\rangle$.
- **Conjugate symmetry**: $\langle x,y\rangle = \langle y,x\rangle$
- **Positive definiteness**: $\langle x,x\rangle \geq o$, with equality if and only if $x=0$ $\|x\|=\sqrt{\langle x,x\rangle}$

The norm induced by the inner product is defined as Consequently, every inner product space becomes a normed vector space, and if the space is complete under this induced norm, it is identified as a Hilbert space.

2.3 Hilbertizable Banach Spaces

A Banach space is said to be **Hilbertizable** if there exists an equivalent inner product norm that makes it isomorphic to a Hilbert space. This is equivalent to the condition that the norm on the Banach space satisfies the **parallelogram identity**: $\|x+y\|^2+\|x-y\|^2=2\|x^2\|+\|y^2\|$ for all $x,y\in X$.

This identity is not only necessary but also sufficient for the existence of an inner product that induces the norm via the polarization identity[2]. In many cases, once such an inner product is defined on a dense subspace, the space can be completed to form a Hilbert space. Thus, the concept of Hilbertizable Banach spaces is central in studying the interplay between purely normed spaces and spaces endowed with richer inner product structures.

2.4 L-Semi-Inner Products

In general Banach spaces, one may not have an inner product available. Instead, a more general notion known as an **L-semi-inner product** is often introduced. An L-semi-inner product preserves many of the properties of an inner product but might not satisfy all the axioms (especially symmetry or linearity in both arguments). Despite these limitations, L-semi-inner products offer a means to relate the norm to algebraic operations and can be used as a stepping stone in understanding when a Banach space admits a full inner product structure, thereby becoming Hilbertizable[1,2].

3. Characterizations of Hilbert Spaces among Banach Spaces

Several key results in functional analysis assist in identifying when a Banach space is isomorphic to a Hilbert space. These characterizations are essential in understanding the possibility and methodology of extending inner product structures.

3.1 Parallelogram Law

The **parallelogram law** serves as the primary criterion for a Banach space to be Hilbertizable. Formally, if a norm $|\cdot|$ on a vector space satisfies $\|x+y\|^2+\|x-y\|^2=2\|x^2\|+\|y^2\|$ for all $x,y\in X$.

then it is possible to define an associated inner product using the polarization identity. For real Banach spaces, the inner product can be recovered as:

$$\langle x,y\rangle=\frac{1}{4}\left(\|x+y\|^2-\|x+y\|^2\right)$$

For complex spaces, this identity adapts to account for complex conjugation. The satisfaction of this law is both necessary and sufficient for the norm to derive from an inner product structure[2].

3.2 Additional Characterizations

Other studies provide alternative perspectives on when a Banach space can be endowed with Hilbert space structure:

- **Kwapień's Characterization**: It has been shown that the validity of a Banach-valued

Parseval's theorem for the Fourier transform characterizes those Banach spaces that are isomorphic to Hilbert spaces. This use of Fourier analysis provides a fascinating bridge between harmonic analysis and functional analysis.

- **Lindenstrauss and Tzafriri's Result**: They demonstrated that a Banach space in which every closed linear subspace is complemented (i.e., every closed subspace is the range of some bounded linear projection) is isomorphic to a Hilbert space. This result underscores the intrinsic geometric structure required for a space to have a Hilbert space equivalent.

- **The Homogeneous Space Problem**: A Banach space that is isomorphic to all its infinite-dimensional closed subspaces has been proven to be isomorphic to a separable Hilbert space. This observation further solidifies the unique nature of Hilbert spaces among Banach spaces in terms of subspace structure and symmetry[2].

These characterizations collectively provide a rigorous framework for determining whether a Banach space is Hilbertizable. Once a space satisfies these conditions, it becomes a candidate for the extension of inner product structures, which is the central theme of this article.

4. Extensions of Inner Product Structures in Hilbertizable Banach Spaces:

4.1 The Core Problem

In many applications, one starts with a Banach space that is not originally defined in terms of an inner product, yet it is known to be Hilbertizable because it satisfies the parallelogram law or one of the other aforementioned characterizations. The core problem this article addresses is: How can an inner product structure be extended from a subspace or via an isomorphic transformation such that the entire space admits an inner product that induces the same topology as its original norm?

4.2 The Role of Isometric Isomorphisms:

If a Banach space ***X*** is Hilbertizable, by definition there exists an isomorphism $T: X \to H$ where ***H*** is a Hilbert space. Such an isomorphism implies that the norm in \$X\$ is equivalent to the norm induced by the inner product of ***H***. Let H. Let $\langle \cdot, \cdot \rangle_H$ denote the inner product on ***H***. Then, we can define an inner product on ***X*** by setting:

$$\langle x, y \rangle x = \langle T(x), Y(y) \rangle H$$

This inner product $\langle \cdot, \cdot \rangle_X$ not only induces a norm equivalent to the original norm of \$X\$ but also extends across the entire space because ***T*** is defined on all of ***X***. In this way, the inner product structure in ***H*** is "transferred" back to **X** through the isomorphism.

4.3 Extension from Subspaces:

Another aspect of the extension problem arises when an inner product is defined on a dense subspace ***Y*** of a Banach space **X**. Since every inner product space (even if incomplete) has a unique Hilbert space completion, one may proceed as follows:

1.Define the Inner Product on the Subspace *Y*: Suppose ***Y*** is equipped with an inner product $\langle \cdot, \cdot \rangle_Y$ which induces the norm . $\| \cdot \|_Y$.

2.Complete the Subspace: Take the completion $\overline{Y}$ of ***Y*** . By construction, is a Hilbert space.

3.Establish an Isomorphism Between *Y* and $\overline{Y}$: If it can be shown that is isomorphic to $\overline{Y}$\overline ***Y*** (or more generally, if is Hilbertizable), then the inner product on ***X*** determines an inner product on by transferring the structure via the isomorphism.This three-step process is fundamentally based on the completion theorem for inner product spaces and shows that the extension of an inner product from a subspace to the entire space is possible provided that the Banach space is Hilbertizable. While this approach is conceptually straightforward, ensuring that the extended inner product induces an equivalent norm to the original remains a non-trivial issue, particularly in spaces where the original norm does not simply come from an inner product.

4.4 Conditions for Consistency and Equivalence

Ensuring that the internal extension of an inner product is consistent with the original Banach space structure involves several considerations:

● **Equivalence of Norms**: The extended inner product must induce a norm that is equivalent to the original norm . Equivalence of norms in finite-dimensional spaces is automatic; however, in infinite-dimensional spaces, careful analysis is required.

● **Continuity of the Extension Map**:

The map that extends the inner product from the subspace ***Y*** to the whole space ***X*** must be continuous with respect to the Banach space topology. This guarantees that the extension respects the convergence structure of ***X***.

● **Maintenance of Completeness**: Since Hilbert spaces are complete with respect to the inner product norm, any extension procedure must ensure that the resulting inner product space remains complete. This might involve taking further completions if the isomorphic image under the extension is not complete. There exist several classical examples where these conditions have been verified. For instance, in the theory of Sobolev spaces, an inner product is often defined on a dense subspace of smooth functions and then extended uniquely to a complete Sobolev space that is Hilbertian when certain conditions on the smoothness parameters are met.

4.5 Theoretical Implications and Mathematical Tools

Several mathematical tools are crucial for analyzing the problem of extending inner product structures:

● **Orthogonal Projections**: In Hilbert spaces, every closed subspace has a unique orthogonal projection. This fact is used to extend inner products from a subspace to the whole space by defining the inner product in terms of projections on complementary subspaces.

● **Polarization Identity**: The polarization identity is used to recover an inner product from a given norm when the parallelogram law holds. This identity not only provides an explicit formula for the inner product but also ensures that the extension is uniquely determined by the underlying norm.

● **Hahn–Banach Theorem**: The non-constructive nature of the Hahn–Banach theorem guarantees the existence of continuous linear functionals and, by extension, tools that allow us to construct L-semi-inner products. Although L-semi-inner products do not always satisfy all properties required for inner products, they are instrumental in understanding when a Banach space may be upgraded to a Hilbert space structure$_2$.

These tools collectively underline the importance of functional analytic methods in understanding and manipulating the inner product structures on Hilbertizable Banach spaces$_2$. The rigorous application of these concepts leads to a broader insight into the geometric and topological structure of infinite-dimensional spaces.

4.6 Summary of the Extension Approach

To summarize the extension process:

* **Step 1:** Identify that the Banach space ***X*** is Hilbertizable, either by verifying the parallelogram law or through other structural characterizations.
* **Step 2:** Determine an appropriate dense subspace ***Y*** of ***X*** where an initial inner product can be defined (or transfer an inner product via an isomorphism to an already established Hilbert space ***X***).
* **Step 3:** Use the polarization identity and completion techniques to extend the inner product from ***Y*** to ***X***.
* **Step 4:** Ensure that the extended inner product induces a norm equivalent to the original norm on ***X***, and verify continuity and completeness.

These steps allow us to conclude that under suitable conditions, the inner product can indeed be extended from a subspace or via an isomorphism, thereby endowing the Banach space with a Hilbert space structure.

5. Examples and Applications:

To better understand the distinctions and relationships between Banach spaces and Hilbert spaces, consider the following table that compares various aspects of these spaces:

5.1 Comparison of Banach and Hilbert Spaces

To better understand the distinctions and relationship between banach spaces and hilbert spaces, consider the following table that compares various aspects of these spaces:

Property	Banach Spaces	Hilbert Spaces
Completeness	Yes (with respect to a norm)	Yes (with respect to the inner product norm)
Norm Induction	Arbitrary norm	Norm i $\|x\| = \sqrt{\langle x, x\rangle}$ (via the formula
Inner Product	Not necessarily	Intrinsic structure (satisfies linearity, symmetry, and available definiteness)
Parallelogram Law	Not generally Satisfied	Satisfied due to inner product structure
Orthogonal Decomposition	Not generally available	Every closed subspace has an orthogonal complement
Extension of Inner	May require additional structure	Already defined by Product the inner product

Table 1: Comparison of Fundamental Properties between Banach and Hilbert Spaces;

This table succinctly outlines the core aspects that distinguish general Banach spaces from Hilbert spaces and provides a clear metric for when a Banach space is Hilbertizable; namely, when its norm satisfies the parallelogram law and an associated inner product structure exists.

5.2 Application in Sobolev Spaces:

In applied mathematics, Sobolev spaces are a classic example of function spaces that are Banach spaces. For certain indices, Sobolev spaces are Hilbert spaces when the underlying norm is derived from an inner product. For example, the Sobolev space , where is an open subset of , can be seen as the completion of the space of smooth functions with compact support under the norm

$$\|u\|H(\Omega) - \left(\int_\Omega |u|^2 + |\nabla u|^2 dx\right)^{1/2}$$

Here, the associated inner product is given by

$$\langle u, v\rangle H^2(\Omega) = \int_\Omega uvdx + \int_\Omega uvdx + |\nabla u|^2 dx$$

This example underscores the natural extension of an inner product from a dense subspace (smooth functions) to the complete Sobolev space, serving as a prototype for more general Hilbertizable Banach spaces.

5.3 Implications in Operator Theory:

The extension of inner product structures within Hilbertizable Banach spaces also plays a significant role in operator theory. Many results in operator theory, such as the existence of adjoint operators and the spectral theorem, depend critically on the inner product structure. For instance, when a Banach space is endowed with an equivalent inner product, several properties of bounded linear operators (including compactness, self-adjointness, and normality) can be studied by transferring the problem to the corresponding Hilbert space. This interplay allows one to apply the rich theory of Hilbert space operators to problems originally posed in a more general Banach setting.

6. Conclusion and Future Directions:

In this article, we have examined the extensions of inner product structures in Hilbertizable Banach spaces. Several key insights have emerged from our analysis:

- **Fundamental Definitions Revisited**: Banach spaces are complete normed vector spaces and do not necessarily possess an inner product, whereas Hilbert spaces are complete with respect to a norm induced by an inner product. Hilbertizable Banach spaces are those that, although defined as Banach spaces, admit an inner product structure that induces an equivalent norm.

● **Characterizations of Hilbertizability**: The parallelogram law is central to identifying when a Banach space can be endowed with an inner product. Additional characterizations using advanced results by Kwapień, Lindenstrauss, and Tzafriri provide further conditions under which a Banach space is equivalent to a Hilbert space.

● **Extension Methodology**:

The extension of inner product structures can be achieved via isomorphisms from the Banach space to a Hilbert space, or by extending an inner product defined on a dense subspace to the entire space. Careful considerations must be made to maintain equivalence of norms, continuity of extensions, and completeness.

● **Practical Illustrations**:

Examples such as Sobolev spaces and applications in operator theory illustrate the significance of these extensions in both theoretical and practical contexts.

The implications of extending inner product structures are far-reaching. They allow for the utilization of geometrical and spectral theory techniques inherent to Hilbert spaces within a broader class of Banach spaces. This synergy not only enriches our understanding of the functional analytic framework but also helps solve complex problems in differential equations, quantum mechanics, and beyond.

Finding Summerized:

- **Verification Criterion**: A Banach space is Hilbertizable if its norm satisfies the parallelogram identity.
- **Extension Procedure**: An inner product defined on a subspace or transferred via an isomorphism can be extended to the whole space, ensuring the resulting inner product induces a norm equivalent to the original.
- **Applications**: Such extensions are beneficial in practical domains such as Sobolev space theory and operator theory, where Hilbert space methods are advantageous.

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Stuty about the zero velocity curves under the influence of Radintion pressure and Poynting- Robertson Drag

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ABSTRACT

In this paper, we have numerically computed the locations of equilbrium point $\mathbf{L_4}$ at different values of $\mathbf{q_{1,}\, q_2\; W_1}$ and $\mathbf{W_2}$ in perturbed cases with respect at the perturbations considered. The effects of all the perturbing parameters are discussed and found considerable deviations in the existing results. It is observed that due to perturbing parameter P-R drags, possible regions for motion the infinitesimal mass become unbounded, which ensure that the equilibrium point is unstable.

Keywords : Study, Zero velocity curves, Radiation pressure, Poynting-Robertson drag.

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1. Introducton :

The angular momentum and the energy integral of the planar three-body problem are used to establish regions of the physical space where motion is allowed to take place. Although forbidden regions exist for both negative and positive values of the energy of the system, the known integrals of the motion always allow for at least one of the three bodies to escape.We conclude that when the energy of the system is negative no bounded motion for all three bodies is assured by the two known integrals. However in the case that one of the members of the system escaeps the other two members will form a binary which may or may not escape. In no case can all the three distances of the bodies increase to infinity when the energy is negative and this, of course, is an evident result.

In 1878 Hill applies the Jacobi constant to describe regions of the possible motion and regions of motion, which is prohibited by separating curves of zero velocity. These zero velocity curves are the principle qualitative part of the restricted three body problem. Zero velocity curves also known as Hill's curves, which discribe the boundary between regions of possible motion and forbidden regions for the motion of the inflmitesimal mass. These curves are the important qualitatiave aspect for the analysis of restricted three body problem. A curve of zero velocity is found by taking a cross-section of the potential surface as a specific energy level with respect to the Jacobi constant of the infinitesimal mass. Depending on the energy of the infinitesimal mass, the topology of a zero velocity curve may change. The Lagrange equilibrium points are clearly located at the critical energy levels, where the topology of the zero velocity curves change. The deep analysis of three dimensional representations of the zero velocity curvs clearly reveals tha the component forces involved in the restricted three body problem, which provide a visual model for understanding the motion of infinitesimal mass near the equilibrium points, Bozis (1976).

In literature, several authors like Kalvouridis (2001). Kumari and Kushvah (2013). etc. have analyzed about zero velocity curves of the infinitesimal mass int he restricted three body problem with or without either radiation prresure or other perturbations. But. no one has studied zero velocity curves the P-R drag. This motivated us to study about the zero velocity curves under the influence of radiation pressure and P-R drag of both the primaries in the restricted three body problem.

Our aim is to examine the motion of the infinitesimal mass in the restricted three body problem using zero velocity curves, when both the primaries are source of radiation with P-R drags. For theat purpose, the potential energy analysis needed to give a conceptual understanding of the resticted three body problem.

In this chapter, zero velocity curves are introduced to obtain a qualitative method of gaining information about the motion without actually solving the differential equations. The Lagrange equilibrium points are special points of zero gradients for the surface. We have obtained the numberical values of the coordinates of the equilibrium points of the problem. Zero velocity curves are obtained from the intersection of the potential of the energy level specified by the Jacobi constant. The curves of zero velocity around equilibrium points are presented, which shows the region of motion of the infinitesimal mass. The effect of the radiation pressure and P-R drag of both primaries plays an important role in analyzing the existence of regions of possible motion for the infinitesimal mass. Reena Kumari and Badam Singh Kushvah (2019) analyzed equilibrium points and zero velocity surface in the restricted four-body problem with solar wind. Recently, Saleem Yousuf and Ram Kishor (2019) studied about the effects of albedo and disc on the zero velocity curves and liner stability of equilibrium points in the generalized restricted three body problem.

The present study deals with the effects of radiation prressure and P-R drag of both the primaries on the regio n of motion of the infinitesimal mass by exploring simulation techniques and drawing the different zero velocity curves. The present chapter contains the following three sections. Numerical computation of equilibrium points are pesented in Section 2. In section 3, we have derived an expression, which is suitable for tracing of different zero velocity curves and last section 4 is devoted for conclusions.

In space dynamics there are a number of systems like two-body, three-body. N-body problem etc. The simplicity and elusiveness of the three-body problem in differeent forms, like the restricted three-body problem (RTBP), restricted four-body problem (RFBP) (which may be considered as an approximation of the two three-body problem) etc, have attracted the attention of researchers for centruries. The motion of a spacecraft or satellite in the Sun-Earth-Moon system is a simple example of RFBP in space. The restricted four-body case has many possible uses in dynamical systems; for example, the fourth body is very useful for saving fuel and time in the trajectory transfers in the restricted four-body problem, Machuy et al (2007).

Few years ago, Burns et al (1979) discussed the radiation forces on small particles in the solar system and examined the different types of effects of the radiating body. However, Schuerman (1980) determined the equilibrium points and P-R drag forces. The dynamical effect of general drag force (i.e., gas drag, nebular drag, P-R drag etc.) in the planar circular restricted three-body problem was discribed by Murray (1994). Also, he has examined the stability of Lagrangian equilibrium points using the linear approximation. Further, Liou et al. (1995) analyzed the effect of radiation pressure, P-R drag, and solar wind drag on the motion of dust grains which are trapped in mean motion resonances with the Sum and the Jupiter in the restricted three-body problem and found that all dust grain orbits are unstable. Again, Kalvouridis et al. (2006) discussed the effect of radiation force due to primaries in the restricted four-body problem using Radzievskii's model and noticed that there are some variations in the result which are unstable for all values of the parameters asssumed by him. Further, Ishwar and Kushvah (2006) and Kushvah (2008), studied the restricted three-body problem with P-R drag and examined the effect of P-R drag force on the motion of infinitesimal body.

The importance of the zero-velocity curves (ZVC) has been outlined by many authors. The investigation of the possible and forbidden regions of motion on the plance, is in many cases the main preliminary step in the study of a new problem. The information gathered in this way concerns not only the regions where the solutions of the dynamicalsystem can exist, but also the locations of the equilibrium points. The subject has been discussed in the past for some of the well-known problems of celestial mechanics such as the three-body problem (Moulton, 1914; Huang, S.1961; Szebehely,1963, 1967; Faubtucg. 1973; Marchal and Saari, 1975; Bozis, 1976; Roy, 1978),

Till now, the researchers pain little attention to the zero-velocity surfaces because of the drawing difficulties. In a very few papers the authors restricted to a qualitative discription of the ZVS without calculating their exact forms and evolution. In some other cases they have used the primitive technique of the superposition of many zero velocity curves to get a draft of 3D representation. The first real interesting effort was made by Lundberg et al., in 1985 in the restricted three-body problem. The 3D pictures presnted in that paper showed clearly the evolution of these surface. The plots, which were produced by using an advanced graphics package, were really impressing. Unfortunately noting more happened on this area ever since.

Our task is to give a global picture of the various ways of eveolution of the regions in space whre the motion of the particle is ppermissible. What we can remark about the influence of the problem parameters in the morphology of the zero-velocity surfaces, can be summarized in the following two propositions. First, the mass parameter mainly infuluences the size of the central and of the peripheral shells. The more larger the parameter is, the more bigger the surface surrounding the central primary and the more smaller the surfaces surrounding the peripheral primarries and second, the parameter has to do with the number of the appearing closed surfaces which enclose the peripheral primaries.

2. Equations of Motion and Equilibrium Points:

The differential equations of motion of infinitesimal mass in the restricted three body problem under the influence of radiation pressure and P-R drag of both the primaries are given.

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}$$

$$y_0 = \pm\left[q_1^{2/3} - \left\{\left(\frac{q_1^{2/3} - q_2^{2/3}}{2} + \frac{1}{2}\right)\right\}\right]^{1/2} \qquad 2.1$$

The co-ordinates of the triangular equilibrium points up to the first-order terms in the perturbing parameters $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{W_1}$ and $\mathbf{W_1}$ provided in our earlier Chapter-2 as the equations (2.3.8) and (2.3.9), which are given as below

$$x^0 = x_0\left\{1 - \frac{\mu\left\{w_2 q_1^{13} \; \frac{w_1(w_2 q_1^{13} - 1)}{2}\right\} + (1-\mu)\left(w_1 q_2^{13} + \frac{w_2(q_1^{13} + q_1^{13} - 1)}{2}\right)}{3\mu(1-\mu)y_0 x_0}\right\}\mu$$

$$\text{where} x_0 = \frac{1}{2} - \mu + \frac{q_1^{1/3} - q_2^{2/3}}{2}$$

$$y^0 = y^0\left\{1 - \frac{\mu\left(w_2 q^{2/3} + \frac{W_1 N_1}{2}\right)\left(q_2^{2/3} - q_1^{2/3} + 1\right) - (1-\mu)\left(W_1 q_2^{2/3} + \frac{W_1 N_1}{2}\right)\left(q_1^{2/3} - q_2^{2/3} + 1\right)}{3\mu(1-\mu)y_0 x_0}\right\}$$

$$\text{and} N_1 = \left(q_1^{2/3} + q_2^{2/3} - 1\right)$$

To know the effect of the perturbing parameters $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{W_1}$ and $\mathbf{W_2}$, on the coordinates of equilibrium points, we have computed the locations of triangular equilibrium points at different values of mass parameter.

For the numbercal computations of equilibrium points, we have chosen dimensionless speed of light $\mathbf{C_d}$=0.26 as well as mass paraameter μ=0.34, and computed the locations of the triangular equilibrium points which are provided.

Table 2.1.

Location of triangular equilibrium points with perturbations

$\mathbf{q_1}$	$\mathbf{q_2}$	$\mathbf{W_1}$	$\mathbf{W_2}$	**X**	**Y**
0.96	0.997	0.039	0.004	0.093307104	0.894621699
0.95	0.997	0.038	0.004	0.092935179	0.891433462
0.94	0.997	0.037	0.004	0.092554181	0.888228667
0.93	0.997	0.036	0.004	0.092163994	0.885006842
0.92	0.997	0.035	0.004	0.091764499	0.881767546
0.98	0.996	0.004	0.001	0.140132685	0.864868359
0.98	0.995	0.004	0.0012	0.138005883	0.864613944
098	0.994	0.004	00014	0.13578929	0.864361011
098	0.993	0004	0.0016	0.133751823	0.864109559
0.98	0.992	0.004	0.0019	0.130477385	0.863825638
0.92	0.996	0.04	0.0019	0.079031436	0.885869059
0.93	0.995	0.039	0.0011	0.080089096	0.885632598
0.94	0994	0038	0.0013	0.81359637	0.888089515
0.95	0.93	0037	0.0017	0.080199829	0.889075345
0.96	0.992	0.036	0.0019	0.081299427	0.886264275
0.97	0.991	0039	0.0021	0076659161	0.895185858

From the Table 2.1, it is observed that triangular point L_1 marginally increases with increase in q_1, q_2.

To know the effects of the perturbing parameters q_1, q_2 W_1 and W_2 on the locatons of triangular equlibrium point, simulations are carried and the results are provided in Figures. 2.1 (a) to 2.6 (a), Te vaiations in X and Y coordinate of trianular equilbrium points L_1 at different values of perturbing parameters q_1, q_2 W_1 and W_2 are plotted in Figures 2.1 (a) to 2.6 (a). From Figure 2.5 (a) it can be observed that the X coordinate of the trinagular equilibrim point L_1 decreascs with increase in perturbng parameters q_1 and q_2, while in Figures 2.5 (a) and 2.6 (a), it is having a marginal decrease with increase in W_1 and W_2, But from Figures 2.5 (b) and 2.6(b), it is observed that Y coordinate of L_1 m is having a very marginal increase with increased radiation parameters q_1 and q_2, P-R drag parametrs W_1 and W_2. From Figure 2.1 (b) , it is noted that in L_4, Y Coordinate decreases.

Description of Figures with Names

Figure 2.1 (a) and (b) : represents X versus q_1 and Y versus q_1,

Respecctivly. (q_2 =0.997, W_1 =0.035, W_2 =0.004)

Figure 2.2 (a) and (b) : represents X versus q_2 and Y versus q_2,

Respecctivly. (q_1 =0.92, W_1 =0.035, W_2 =0.004).

Figure 2.3 (a) and (b) : represents X versus W_1 and Y versus W_1,

Respecctivly. (q_1 =0.92, q_2 =0.997, W_2 =0.004).

Figure 2.4 (a) and (b) : represents X versus W_2 and Y versus W_1,

Respecctivly. (q_1 =0.92, q_2 =0.997, W_2 =0.035).

Figure 2.5 (a) and (b) : W_1 =0.035,, W_2 =0.004,

Figure 2.6 (a) and (b) here : q_1 =0.092,, q_2 =0.997,

3. Zero Velocity Curves:

In order to discuss the different zero velocity curves of the infinitesimal mass in the restriced three-body prolem with rationa pressures and P-R drags of both the primaries, multiplying the first equatin of 2.1 by $2\dot{x}$ and the second equation by $2\dot{y}$ and adding both, we get;

$$2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} = 2\dot{x}\frac{\mu U}{\partial x} + 2\dot{y}\frac{\partial U}{\partial y} \qquad \textbf{3.1}$$

The above equaton (4.3.) can be written as.

$$\frac{d}{dt}\left[\dot{x}^2 + \dot{y}^2\right] = 2\frac{dU}{dt} \qquad \textbf{3.2}$$

Integrating the equation (4.3.2) with respect to the time t, we get

$$\dot{x}^2 + \dot{y}^2 = 2U + C \qquad \textbf{3.3}$$

where C is the constant of integration and known as Jacobi constant.

Moulton, (1914); Murray and Dermott, (2000).

Since, the potential function of the problem is written by:

$$U = \frac{x^2+y^2}{2} + \frac{(1-\mu)q_1}{r^1} + \frac{\mu q_2}{r_2}\left[\frac{(x+\mu)\dot{x}+y\dot{y}}{2r_1^2} - \arctan\left(\frac{y}{x+\mu}\right)\right]$$

$$+W_2\left[\frac{(x+\mu-1)\dot{x}+y\dot{y}}{2r_2^2} - \arctan\left(\frac{y}{x+\mu-1}\right)\right] \qquad \textbf{3.4}$$

Therefore, we obtain the Jacobi constant C using the equatios 3.3 and 3.4 follows:

$$C = x^2 + y^2 - \dot{x}^2 - \dot{y}^2 + \frac{2(1-\mu)q_1}{r_1} + \frac{2\mu l_2}{r_2}$$

$$+W_1\left[\frac{x+\mu)\dot{x}+y\dot{y}}{r_1^2} - 2\arctan\left(\frac{y}{x+\mu}\right)\right]$$

$$+W_2\left[\frac{x+\mu)\dot{x}+y\dot{y}}{r_2^2} - 2\arctan\left(\frac{y}{x+\mu-1}\right)\right] \qquad \textbf{3.5}$$

That means the equation 3.5 becomes zero velocity surface , when its velocity terms become zero. e.e., when $\dot{x} = \dot{y} = 0$ Kushvah et al (2012), Saleem and Kishor (2019), which is given as below.

$$C = x^2 + y^2 + \frac{2(1-\mu)q_1}{r^1} + \frac{2\mu l_2}{r^2} - 2W_1$$

$$-W_2 \arctan\left(\frac{y}{x+\mu-1}\right) \qquad \textbf{3.6}$$

From the equation 3.6 we have obtained, various zero velocity curves by putting different numerical value for C. The zero velocity curves are now pulsating with frequency of the nominal motion. Hence, in the planar restricted three body problem, the different zero velocity curves are obtained from the equation 3.6 for the infinitesimal mass around the binary system, at various values of the perturbing parameters.

We have traced different zero velocity curves of the infinitesimal mass, with th energy level for different values C such as C_1=3.9132, C_1=3.8495, C_3=3.7221, and C_5=3.6584, and explain the region from which the infinitesimal mass is, called the excluded region. The excluded region, which evolves as the value of C is varied. Any point, which are not in theexcluded region is called allowed region. The zero velocity curves with respect to C_1 C_2 C_3 C_4 and C_5 are plotted and presented in the following Figures 3.1 to 3.12, it is noticed that the possible region for the motion of infinitesimal mass becomes, unbounded, whch signals the instability of all equilibrium points.

Figure discription

(3.1) : Zero velocity curves when Jacobi constant C_1-3.9132 where W_1=W_1=0

(3.2) : Zero velocity curves when Jacobi constant C_2-3.8495where W_1=W_1=0

(3.3) : Zero velocity curves when Jacobi constant C_3-3.7221 where W_1=W_1=0

(3.4) : Zero velocity curves when Jacobi constant C_4-3.7221 where W_1=W_1=0

(3.5) : Zero velocity curves when Jacobi constant C_5-3.6584 where W_1=W_1=0

(3.6) : Zero : Combination of (3.1) to (3.5)

(3.7) : Zero velocity curves when Jacobi constant C_1-3.9132 where W_1=0.035, W_1=0.004

(3.8) : Zero velocity curves when Jacobi constant C_2-3.8495 where W_1=0.035, W_1=0.004

(3.9) : Zero velocity curves when Jacobi constant C_3-3.7858 where W_1=0.035, W_1=0.004

(3.10) : Zero velocity curves when Jacobi constant C_4-3.7221 where W_1=0.035, W_1=0.004

(3.11) : Zero velocity curves when Jacobi constant C_1-3.9132 where W_5=0.035, W_1=0.004

(3.12) : Combinatiion of figure 3.7 to figure 3.11

Description of the Results:

We have constructed the plots of zero velocity curves for the mass parameter μ=0.34 in Fig. 3.1 to Fig. 3.12. The Jacoby constants using the triangular equilibrium points are calculted in the absence of W_1 and W_2 wich q_1 =q_2 = 1.0,0.95,0.90,0.85, and 0.80 and their correspoinding Jacob constants are C_1 =3.9132, C_2= 3.8495, C_3 = 3.7858, C_4 =3.7221 and C_5 =3.6584, respectively. Clearly, each plot displays two distinct curves one situated in closer proximity to the primaries, and the other positioned at a greater distance from the primaries. We use the terms inner and outer curves for he sake of convenience. It can be seen that with decreasing the value of the Jacobi constant, the inside curves of zero velocity become more oval-like structures, and then after a particular value of the Jacobi constant, the two inside curves merge together, and these from a dumbell-like structure (Fig. 3.5, Fig. 3.10 and Fig. 3.11) A similar pattern can be observed in the shape of the zero velocity cuves when the perturbation due to the drage forces W_1 and W_2 are considered which are depicted in fig. 3.7 to Fig. 3.12. Moreover. some cuts on the zero velocity curves are observed due to the singularity in the potential function. The cuts on the zero velocity curves are near the primaries in such a way that the set of cuts is along a straight line parallel to the y-axis, passing through the centre of the primaries. Here, the cuts refer to the region where the motion of the particle is possible.

4. Conclusions:

In this Chapter, we have numberically computed the locations of triangular equilibrium point L_1 at different values of q_1, q_2, W_1, and W_2 in perturbed cases with respect to the perturbations considered in the present study. The effects all the perturbing parameters are discussed and found considerable deviatons in the existing results. The zero velocity curves and obtained corresponding to equilibrium point qL_1, It is observed that due to the perturbing parameter P-R drags, possible regions for motion of the infinitesimal mass bcome unbounded, which ensure that the equilibrium point is unstable.

The present analysis carried out in this thesis is restricted up to first order effect of the perturbating parameters solar radiation pressure and P-R drag of both the primaries, which will be useful to learn more about the global problem. The results obtained may be helpful to analyze the more global problems under the presence of other perturbations such as albedo solar wind drag, Stokes drag etc. The present study is limited up to first order terms of the perturbing parameters, which may be extended to include higher order terms. Also it is an open problem to consider normalization to sixth order Hamiltonian to apply KAM theorem to examine stability of the problem.

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